MATH 54 – MOCK MIDTERM 3 – SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) **TRUE** If A is similar to B, then $det(A) = det(B)$

(If A is similar to B, then $A = PBP^{-1}$ for some P, so $det(A) = det(PBP^{-1}) = det(P)det(B)det(P^{-1}) = det(B)$)

(b) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 4, 0$, then A is diagonalizable

(A is a 3×3 matrix with 3 *distinct* eigenvalues)

(c) **FALSE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 4, 0$, then A is invertible

(0 is an eigenvalue of A, so by the IMT A is not invertible)

(d) **TRUE** If A is a 4×4 matrix with eigenvalues $\lambda = 1, 2, 2, 3$, then $det(A) = 12$

($det(A)$ is the product of the eigenvalues of A, including multiplicities, so here $det(A) = 1 \times 2 \times 2 \times 3 = 12$)

Date: Friday, July 27th, 2012.

(e) **FALSE** If λ is an eigenvalue of A, then $Nul(\lambda I - A)$ *could* be $\{\mathbf{0}\}$.

(*by definition* an eigenvalue λ is a number such that $A\mathbf{v} = \lambda\mathbf{v}$, that is $(A - \lambda I)\mathbf{v} = \mathbf{0}$, hence $Nul(\lambda I - A) \neq \{\mathbf{0}\}$ because it contains $\mathbf{v}$)

2. *(10 points)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

This means:

- If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

- If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

(a) **TRUE** If A is diagonalizable, then A^2 is diagonalizable

Since A is diagonalizable, there exists a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. But then:

$$A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$$

And this is of the form $\widetilde{P}\widetilde{D}\widetilde{P}^{-1}$, where $\widetilde{P} = P$ and $\widetilde{D} = D^2$, which is diagonal!

(b) **FALSE** If A has only one eigenvalue, then A is not diagonalizable

For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It is diagonalizable **because it is a diagonal matrix**, but it only has one eigenvalue, $\lambda = 1$.

3. *(30 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 8 & -4 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Eigenvalues:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 & 0 & 0 \\ 0 & \lambda - 3 & 0 & 0 \\ -8 & 4 & \lambda - 7 & 0 \\ 0 & 0 & 0 & \lambda - 7 \end{vmatrix} = (\lambda - 3)(\lambda - 3)(\lambda - 7)(\lambda - 7) = (\lambda - 3)^2(\lambda - 7)^2 = 0$$

Which gives $\boxed{\lambda = 3, 7}$

Eigenvectors:

$\underline{\lambda = 3}$:

$$Nul(3I - A) = Nul \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 4 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} = Nul \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

But then $2x - y + z = 0$ and $t = 0$ gives $x = \frac{y}{2} - \frac{z}{2}$ and $t = 0$, so:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{y}{2} - \frac{z}{2} \\ y \\ z \\ 0 \end{bmatrix} = y \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence:

$$Nul(3I - A) = Span \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}$$

$\underline{\lambda = 7}$:

$$Nul(7I - A) = Nul \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ -8 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

But then $x = 0$ and $y = 0$, so:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \\ t \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence:

$$Nul(7I - A) = Span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

<u>Answer:</u>

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: Another acceptable answer is:

$$D = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

4. *(30 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$det(\lambda I - A)) = \begin{vmatrix} \lambda - 3 & -1 & 0 \\ -1 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)((\lambda - 3)^2 - 1) = (\lambda - 2)(\lambda - 2)(\lambda - 4) = (\lambda - 2)^2(\lambda - 4)$$

Which gives $\boxed{\lambda = 2, 4}$

<u>Eigenvectors:</u>

$\underline{\lambda = 2}$

$$Nul(2I - A) = Nul \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x + y = 0$, so $y = -x$, and so:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And so:

$$Nul(2I - A) = Span \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 4}$

$$Nul(4I - A) = Nul \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Nul \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x = y$, $z = 0$, and so:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

And so:

$$Nul(4I - A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

<u>General Solution:</u>

$$\mathbf{x}(t) = Ae^{2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + Be^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + Ce^{4t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

5. *(20 points, 10 points each)*

(a) *(10 points)* Use **undetermined coefficients** to find the general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Note: You may use the fact that the general solution to $\mathbf{x}' = A\mathbf{x}$ is: $\mathbf{x}_0(t) = Ae^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

<u>Particular solution:</u> Usually you would guess y_p to be a constant a, so here:

Guess $\mathbf{x}_p(t) = \mathbf{a} = \begin{bmatrix} A \\ B \end{bmatrix}$

Plug this into $\mathbf{x}_p' = A\mathbf{x}_p + \mathbf{f}$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

which gives us:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A+2B+1 \\ 3B+3 \end{bmatrix}$$

Which gives us: $A+2B+1=0$ and $3B+3=0$.

Solving this system of equations (either by using linear algebra or doing it directly), we get $A=1$, $B=-1$, and so:

$$\mathbf{x}_p(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

General solution:
Using this and the **Note** at the beginning, we get:

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{x}_p(t) = \mathbf{x}_0(t) = Ae^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) *(10 points)* Use **variation of parameters** to find the general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} e^{2t} \\ e^{4t} \end{bmatrix}$$

Note: You may use the fact that the general solution to $\mathbf{x}' = A\mathbf{x}$ is: $\mathbf{x}_0(t) = Ae^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Particular solution:
Using the **Note**, we get that the Wronskian matrix is:

$$\widetilde{W}(t) = \begin{bmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

Now suppose $\mathbf{x}_p(t) = v_1(t) \begin{bmatrix} e^t \\ 0 \end{bmatrix} + v_2(t) \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$, then use the variation of parameters formula:

$$\widetilde{W}(t)\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \mathbf{f}(t)$$

Here we get:

$$\begin{bmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{4t} \end{bmatrix}$$

So:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{bmatrix}^{-1}\begin{bmatrix} e^{2t} \\ e^{4t} \end{bmatrix}$$

But

$$\begin{bmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{bmatrix}^{-1} = \frac{1}{e^{4t}}\begin{bmatrix} e^{3t} & -e^{3t} \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ 0 & e^{-3t} \end{bmatrix}$$

(here we used the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

Hence:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^t - e^{3t} \\ e^t \end{bmatrix}$$

Hence $v_1'(t) = e^t - e^{3t}$, so $v_1 = e^t - \frac{1}{3}e^{3t}$, and $v_2'(t) = e^t$, so $v_2(t) = e^t$ and finally:

$$\begin{aligned}
\mathbf{x}_p(t) =& v_1(t)\begin{bmatrix} e^t \\ 0 \end{bmatrix} + v_2(t)\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \\
=& \left(e^t - \frac{1}{3}e^{3t}\right)\begin{bmatrix} e^t \\ 0 \end{bmatrix} + e^t\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \\
=& \begin{bmatrix} e^{2t} - \frac{1}{3}e^{4t} \\ 0 \end{bmatrix} + \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} \\
=& \begin{bmatrix} e^{2t} + \frac{2}{3}e^{4t} \\ e^{4t} \end{bmatrix}
\end{aligned}$$

General solution:

$$\begin{aligned}\mathbf{x}(t) =& \mathbf{x}_0(t) + \mathbf{x}_p(t)\\ =& Ae^t \begin{bmatrix}1\\0\end{bmatrix} + Be^{3t}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}e^{2t} + \frac{2}{3}e^{4t}\\ e^{4t}\end{bmatrix}\end{aligned}$$